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A PROBABILISTIC THEORY OF UTILITY

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A PROBABILISTIC THEORY OF UTILITY

by

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A PROBABILISTIC THEORY OF UTILITY[1]

R. Duncan Luce

1. Introduction

The mathematical formulation of individual decision making - utility theory - has been traditionally based upon non-probabilistic preference relations, usually postulated to be weak orders. Few authors have been satisfied with the assumption that preference is transitive, which is easily demonstrated to be at variance with fact. Yet this assumption has been retained as an approximation to reality because of its nice mathematical properties; for example, only such a preference relation can be represented by a single order preserving numerical function. Nonetheless, a number of people have voiced a desire for a probabilistic theory, mainly, I would judge, so as to be able better to handle empirical data. Here is an attempt at such a theory. One pleasing aspect of this theory is that it seems to have conceptual import as well as giving the empiricist a more manageable tool.

The intuitive idea behind the mathematical framework I shall present is this: Pairs of elements (or alternatives or stimuli) are selected from a given set S, and a person is required to choose from each pair the alternative which he views as "superior" according to some given comparative dimension -- a dimension whose choice depends upon the particular empirical context. It may be preference, intensity, size, loudness, importance, etc. It will be

[1] This paper completely supercedes "Two results on semi-ordered mixture spaces," Technical Report no. 13, Behavioral Models Project, Bureau of Applied Social Research, Columbia University.

convenient to think of the underlying comparative dimension as "strict preference," but it must be kept in mind that this is only one of the many possible interpretations which can be given to the formalism.

If a and b are elements of S, it is postulated that there exists a probability P(a,b) that a is judged as strictly preferred to b. One problem to be faced is the axiomatic formulation of the fact that many dimensions seem to impose something like a linear ordering upon the underlying set S. Once this is done, S is specialized to be a set of risky alternatives of the form aαb, where this symbol is interpreted to mean that alternative a arises with probability α and alternative b with probability 1-α. The exact meaning that should be given to the word "probability" will receive some attention. At this point, my central assumption will be introduced, namely, that the activity of deciding which of two alternatives is preferred is statistically independent of the activity of discriminating which of two probabilities is larger. Indeed, this entire paper can be described as exploring the consequences of this one assumption.

It may intrigue the reader to know in advance some of the <u>interpretive</u> conclusions we shall reach.

i. If the independence assumption is met, then the probabilities entering into the risky alternatives <u>must be</u> subjective probabilities, where subjective probability is given a well defined and operational meaning which is identical to a traditional usage of "subjective" in psychophysics.

ii. If the independence assumption is met, then the mathematical form of the discrimination function for subjective probabilities is completely determined up to a single parameter.

iii. If the independence assumption is met and if a certain natural weak order induced by the probabilities P can be represented by a linear utility function, then that function is a subjective sensation scale in the sense used in psychology; and, under slightly stronger assumptions, the converse is true: a sensation scale is a linear utility function.

iv. Suppose the basic alternatives in a situation are sums of money and that there are at least three different sums such that for any pair of these sums a particular person will invariably prefer the larger, everything else being held equal. If his discrimination of subjective probabilities is not perfect and if the independence assumption is met for gambles of money, then there cannot exist a utility function for these sums of money such that the utility of a gamble is equal to the expected utility of its components. Put another way, if his preferences among these money gambles can be represented by a linear utility function, then his preference discrimination cannot be independent of his subjective probability discrimination , unless the latter is perfect.

Background references for this paper are Edwards [1], Luce [4], Luce and Edwards [5], and von Neumann and Morgenstern [6].

2. Discrimination and Linear Structures

Definition 1. Given a set S, a real-valued function P with domain S×S is called a discrimination structure on S provided that:

D1. $P(a,b) \geq 0$, for all $a,b \in S$,

D2. $P(a,b) + P(b,a) \leq 1$, for all $a,b \in S$,

and D3. there exist $a^*,b^* \in S$ such that $P(a^*,b^*) \neq P(b^*,a^*)$.

A discrimination structure is called reflexive if

D4. $P(a,a) = 0$, for all $a \in S$.

Since $P(a,b)$ is interpreted as the probability that a is strictly preferred to b, $1-P(a,b)-P(b,a)$ is interpreted as the probability that a is judged "indifferent" to b.

Definition 2. If P is a discrimination structure on S, the binary relation $\gtrsim$ on S is defined as: $a \gtrsim b$ if $P(a,c) \geq P(b,c)$ and $P(c,b) \geq P(c,a)$ for all $c \in S$. If $a \gtrsim b$ and $b \gtrsim a$, then $a \sim b$ is written. If $a \gtrsim b$ and not $a \sim b$, then $a > b$ is written.

Theorem 1. If P is a discrimination structure on S, then $\gtrsim$ is a quasi ordering of S.

Proof. Obvious.

Definition 3. A discrimination structure P on S is called transitive if $a > b$ implies $P(b,a) = 0$.

If the preferences registered by people are assumed to be governed by a discrimination structure, then in general a sample of preferences will include intransitivities.

Any reflexive discrimination structure is transitive, for if $a > b$, then $0 = P(a,a) \geq P(b,a) \geq 0$.

There is nothing in the definition of a discrimination structure which attempts to capture, even in probabilistic terms, the idea that a preference dimension imposes a linear, or weak, ordering on S. The appropriate restriction is suggested by definition 2, but it can also be arrived at by intuitive considerations such as these: Suppose that it is possible to string the elements of S out as a linear array in such a manner that, for each $a \in S$, $P(x,a)$ is an increasing function of x and $P(a,x)$ is a decreasing function of x, then we may think of this array as reflecting increasing preference. Thus, we are led to:

Definition 4. A discrimination structure P on S is called a linear structure provided that for every $a,b \in S$ either

i. $P(a,c) \geq P(b,c)$ and $P(c,b) \geq P(c,a)$, for all $c \in S$,

or ii. $P(a,c) \leq P(b,c)$ and $P(c,b) \leq P(c,a)$, for all $c \in S$.

Theorem 2. If P is a linear structure on S, then $\succsim$ is a weak ordering of S.

Proof. Obvious.

If a linear structure is reflexive, and so transitive, then no intransitivities, but in general this is not so. sample of reported preferences will include strict preference / I propose that decision theories postulate the existence of linear structures, not weak orders, as the basic preference information. Empirically, observations on subjects would be used, not to construct weak orders directly, but to estimate the underlying linear structure. Once that is recovered, definition 2 generates a weak order suitable for use in the present mathematical theories. In much of the following work certain assumptions about linear structures will be made and their consequences for the induced weak orders explored. For the most part we shall be concerned with underlying spaces consisting of risky alternatives and with weak orders that can be represented by linear utility functions. It will be shown, among other things, that certain plausible conditions on a linear structure imply that its weak order can be represented by a linear utility function and that the entire linear structure can be represented by the utility function plus one other real-valued function of a real variable.

It should be pointed out that the present concepts provide not only a framework for a probabilistic theory of utility, but also include as special cases all the mathematical models of discrimination studied in psychophysics. In that work, S is taken to be the positive real numbers. While I have no intention of examining the psychophysical model here, I shall note from time to time close conceptual relations between it and this utility model. Much of this work was suggested by a knowledge of both the classical discrimination model and the von Neumann utility theory [6], and, whether we like it or not, some of the consequences of a probabilistic utility theory are psychophysical in nature. The reader interested in the formal mathematical structure of the psychophysical model can consult [5].

One thing that is clear is that few results can be obtained about anything so general as a discrimination structure, that to find any non-trivial consequences it is necessary to postulate an underlying set S which is dense and which has considerable mathematical structure. For example, in psychology S is taken to be the positive real numbers; here it will be taken to be a somewhat weakened form of the notion of a mixture space. It is plausible that a great many results, possibly including both mine and those of psychophysics, can be obtained by choosing S to be an appropriate topological space, but so far no work in this direction has been undertaken.

3. Representations of Linear Structures by Semiorders

Before specializing S to a space of risky alternatives, the relation between linear structures and a purely algebraic model of discrimination [4] will be examined. An algebraic representation of any discrimination structure can be obtained via the following standard psychophysical device:

Definition 5. Let P be a discrimination structure on S and let k be a number, $\frac{1}{2} \leq k < 1$, then the pair of binary relation (L_k, I_k) on S is defined by:

$$aL_kb \quad \text{if} \quad P(a,b) > k$$
$$aI_kb \quad \text{if} \quad P(a,b) \leq k \text{ and } P(b,a) \leq k.$$

It will be recalled that in [4] the following concept was introduced:

Definition 6. A pair of binary relations (L,I) on a set S is called a *semiorder* if for every $a,b,c,d \in S$ the following axioms are met:

S1. exactly one of aLb, bLa, or aIb obtains,

S2. aIa,

S3. aLb, bIc, cLd imply aLd,

S4. aLb,bLc,bId imply not both aId and dIc.

<u>Theorem 3. The relations</u> (L_k, I_k) <u>of a linear structure form a semiorder.</u>

Proof. S1. Axiom D2 of Definition 1 and $k \geq 1/2$.

S2. By axiom D2. $P(a,a) \leq 1/2 \leq k$, so $aI_k a$.

S3. Suppose $aL_k b$, $bI_k c$, $cL_k d$. We show $a \gtrsim c$. Suppose $c > a$, then by definition 2, $P(c,b) \geq P(a,b)$, but by definition 5, $k \geq P(c,b)$ and $P(a,b) > k$, which is impossible. Thus, by theorem 2, $a \gtrsim c$. Then by definition 2, $P(a,d) \geq P(c,d) > k$, so $aL_k d$.

S4. Suppose $aL_k b$, $bL_k c$, $bI_k d$. Since $P(a,b) > k \geq P(b,b)$, definitions 2 and 4 imply $a > b$. In like manner, $b > c$. As above, one shows that either $a \gtrsim d \gtrsim b$ or $b \gtrsim d \gtrsim c$. If the former, $P(d,c) \geq P(b,c) > k$, so not $dI_k c$. If the latter, not $aI_k d$.

We observe that if a linear structure is both reflexive and transitive, then the range of k may be extended to $0 < k < 1$.

Given a pair of relations (L, I), another pair of relations $(>', \sim')$ can be defined:

$a >' b$ if either i. aLb,

ii. aIb and there exists c such that aIc and cLb,

iii. aIb and there exists d such that aLc and cIb.

$a \sim' b$ if neither $a >' b$ nor $b >' a$.

In [4] it was shown that if (L, I) is a semiorder, then $(>', \sim')$ is a weak order, which we speak of as the weak order induced by the semiorder.

<u>Theorem 4. If</u> P <u>is a linear structure on S,</u> k <u>a number,</u> $1/2 \leq k < 1$, $\gtrsim$ <u>the weak order induced by</u> P <u>according to definition</u> 2, <u>and</u> $\gtrsim_k$ <u>the the weak order induced by the semiorder</u> (L_k, I_k), <u>then</u> $>_k$ <u>implies</u> $>$.

Proof. If $a >_k b$, then either

i. aL_kb, so $P(a,b) > k \geq P(a,a)$, so by definitions 2 and 4, $a > b$;

or ii. aL_kb, aL_kc, and cL_kb, so $P(c,b) > k \geq P(c,a)$, so $a > b$;

or iii. aL_kb, aL_kd, and dL_kb, so $P(a,d) > k \geq P(b,d)$, so $a > b$.

4. Decomposable Discrimination Structures on Weak Mixture Spaces

Much of modern utility theory treats an underlying space of risky alternatives $a\alpha b$, where this symbol is usually interpreted to mean that one, but not both, of the alternatives a or b results, the former with probability α and the latter with probability $1-\alpha$. Just what interpretation should be given to the word "probability" is a point of dispute; we will be led to a particular subjective probability concept in section 7. However, at present the concept of probability need not be brought in at all, but rather α can be taken to denote the occurrence of a well specified event, such as whether a given die thrown by a particular mechanism at a specified time comes up six, or whether the word "Britain" will be found in column 5, page 2 of tomorrow's New York Times. If α denotes the occurrence of a particular event, let $\bar{\alpha}$ denote its non-occurrence. In these terms, $a\alpha b$ will be interpreted to mean that a results if event α occurs and b if it does not. The set of events which will be admitted experimentally will have to satisfy certain special properties in the light of the definitions we shall introduce, namely: basic events shall be independent of one another and there is at least one event, o, which has (subjective) probability 0. The axioms given below, and in later sections, are closely related to those given by Hausner [3] for what he called a mixture space; however, even when we introduce numbers we shall not need all of his axioms.

Definition 7. Let E be a Boolean algebra with null element o. A set S is called a weak mixture space over E if for all $a,b \varepsilon S$ and $\alpha \varepsilon E$,

R1. $a\alpha b \varepsilon S$,

R2. $a\alpha a = a$,

R3. $a\alpha b = b\bar{\alpha}a$

R4. $aob = b$.

The next concept, which has no analogue in non-probabilistic utility theory, is crucial throughout the rest of this paper. Consider the two alternatives $a\alpha b$ and $a\beta b$. It is plausible that the former will be preferred to that latter if and only if either

i. alternative a is preferred to b and event α is perceived as more probable than event β,

or ii. alternative b is preferred to a and event β is perceived as more probable than event α.

If we assume that there is a probability $Q(\alpha,\beta)$ that α is discriminated as more probable than β and if we assume that the perception of preference is statistically independent of the discrimination of relative probability magnitudes, then there is (see Definition 8) a simple expression for the probability that $a\alpha b$ is preferred to $a\beta b$. It is true that there is some evidence which suggests that these processes may not be statistically independent, at least when α and β are identified with their objective probabilities; however, this is far from certain at present. In any event, it is interesting to examine the consequences of this assumption -- especially since some of them are intimately related to the notion of a linear utility function.

Definition 8. A discrimination structure P on a weak mixture space S is said to be decomposable if there exists a real valued function Q on ExE, which is called the core of P, such that for all $\alpha,\beta\varepsilon E$,

i. $Q(\alpha,\beta) \geq 0$,

ii. $Q(\alpha,\beta) + Q(\beta,\alpha) \leq 1$,

and iii. for all $a,b\varepsilon S$,

$$P(a\alpha b, a\beta b) = P(a,b)Q(\alpha,\beta) + P(b,a)Q(\beta,\alpha).$$

It is simple to give an expression for Q in terms of P, namely:

$$Q(\alpha,\beta) = \frac{P(a\alpha b, a\beta b)P(a,b) - P(a\beta b, a\alpha b)P(b,a)}{P(a,b)^2 - P(b,a)^2},$$

for any $a,b\varepsilon S$ such that $P(a,b) \neq P(b,a)$ (by definition 1, there is at least one such pair). This formula renders it possible to estimate Q from empirical preference data, if the assumption is correct, and to determine whether it is correct by holding α and β fixed while varying a and b.

At this point I could offer an example of a decomposable discrimination structure, but since later results will make constructing examples completely trivial, there is little point in taking the space here.

The semiorder representation of linear structures (see the previous section) is inadequate for the study of decomposable linear structures for the following reason: If a probability cutoff k is chosen and it is applied to both P and Q, then either $\alpha I_k \beta$ or $aI_k b$ imply $(a\alpha b)I_k(a\beta b)$, for in the former case

$$\begin{aligned} P(a\alpha b, a\beta b) &= P(a,b)Q(\alpha,\beta) + P(b,a)Q(\beta,\alpha) \\ &\leq k[P(a,b) + P(b,a)] \\ &\leq k, \end{aligned}$$

and similarly $P(a\beta b, a\alpha b) \leq k$. In the latter case a similar proof holds. However, the converse need not be true. Consider, for example, a case where $k = 5/8$, $P(a,b) = 3/4$, $P(b,a) = 1/4$, $Q(\alpha,\beta) = 3/4$, and $Q(\beta,\alpha) = 1/4$, then it follows that $P(a\alpha b, a\beta b) = 5/8$ and $P(a\beta b, a\alpha b) = 3/8$, so $(a\alpha b)I_k(a\beta b)$ even though neither $aI_k b$ nor $\alpha I_k \beta$.

Definition 9. Let E be a Boolean algebra. A real-valued function f on ExE is called symmetric if $f(\alpha,\beta) = f(\bar{\beta},\bar{\alpha})$.

Definition 10. A discrimination structure P on S is called additive if there exists a constant K, $0 < K \leq 1$, such that for all $a,b \varepsilon S$, $P(a,b) + P(b,a) = K$.

Theorem 5. The core of a decomposable discrimination structure on a weak mixture space is itself a discrimination structure, it is symmetric, and it is either reflexive or additive with K = 1.

Proof. Let P be the decomposable discrimination structure and Q its core. To show Q is a discrimination structure it is sufficient to show $Q(e,o) \neq Q(o,e)$, where we call $\bar{o} = e$.

Suppose not, then for any $a,b \varepsilon S$, axioms R3 and R4 imply

$$\begin{aligned} P(a,b) &= P(aeb, aob) \\ &= P(a,b)Q(e,o) + P(b,a)Q(o,e) \\ &= [P(a,b) + P(b,a)]Q(e,o). \end{aligned}$$

By axiom D3 of definition 1, there exist $a^*, b^* \varepsilon S$ such that $P(a^*,b^*) \neq P(b^*,a^*)$, but from what has just been shown,

$$P(a^*,b^*) = [P(a^*,b^*) + P(b^*,a^*)]Q(e,o)$$
$$= [P(b^*,a^*) + P(a^*,b^*)]Q(e,o)$$
$$= P(b^*,a^*),$$

which is a contradiction.

To show that Q is symmetric, let a^* and b^* be the elements described in axiom D3 (definition 1), then by axiom R3 (definition 7)

$$P(a^*\alpha b^*, a^*\beta b^*) = P(b^*\bar{\alpha}a^*, b^*\bar{\beta}a^*),$$

for all $\alpha,\beta \varepsilon E$. Apply the decomposition assumption to both sides,

$$P(a^*,b^*)Q(\alpha,\beta) + P(b^*,a^*)Q(\beta,\alpha)$$
$$= P(b^*,a^*)Q(\bar{\alpha},\bar{\beta}) + P(a^*,b^*)Q(\bar{\beta},\bar{\alpha}).$$

Collecting terms,

$$P(a^*,b^*)[Q(\bar{\beta},\bar{\alpha}) - Q(\alpha,\beta)] + P(b^*,a^*)[Q(\bar{\alpha},\bar{\beta}) - Q(\beta,\alpha)] = 0$$

Interchange the roles of a^* and b^*,

$$P(b^*,a^*)[Q(\bar{\beta},\bar{\alpha}) - Q(\alpha,\beta)] + P(a^*,b^*)[Q(\bar{\alpha},\bar{\beta}) - Q(\beta,\alpha)] = 0$$

This pair of equations has no non-trivial solution since the determinant of coefficients, $P(a^*b^*)^2 - P(b^*,a^*)^2$, is non-zero by the choice of a^* and b^*, so Q is symmetric.

Using axiom R2, and the decomposition assumption, then for any $a \varepsilon S$ and any $\alpha \varepsilon E$,

$$P(a,a) = P(a\alpha a, a\beta a) = P(a,a)[Q(\alpha,\beta) + Q(\beta,\alpha)]$$

If $P(a,a) > 0$ for any a, then $Q(\alpha,\beta) + Q(\beta,\alpha) = 1$, so Q is additive with $K = 1$. If P is reflexive, then by axioms D1 and D3, there exist $a^*, b^* \varepsilon S$ such that $P(a^*,b^*) + P(b^*,a^*) > 0$,

so

$$P(a^*\alpha b^*, a^*\alpha b^*) = 0 = P(a^*,b^*)Q(\alpha,\alpha) + P(b^*,a^*)Q(\alpha,\alpha)$$

implies $Q(\alpha,\alpha) = 0$, i.e., Q is reflexive.

Definition 11. A discrimination structure P on a weak mixture space is called *regular* if for all $a,b\varepsilon S$ and $\alpha\varepsilon E$, $P(a,a\alpha b) = P(b\alpha a,b)$.

The importance of the concept of regularity will not become apparent until section 6.

Theorem 6. *A decomposable discrimination structure P on a weak mixture space is regular and either it is reflexive and transitive or it is additive. It is reflexive if and only if its core is reflexive. It is additive if and only if its core is additive.*

Proof. Using axiom R4 and the decomposability assumption,

$$P(a,a\alpha b) = P(aeb,a\alpha b) = P(a,b)Q(e,\alpha) + P(b,a)Q(\alpha,e)$$

$$P(b\alpha a,b) = P(b\alpha a,bea) = P(b,a)Q(\alpha,e) + P(a,b)Q(e,\alpha),$$

and so P is regular.

From the proof of theorem 5, we know that Q is reflexive if and only if P is reflexive. By the remark following definition 3, P is transitive. If P is not reflexive, choose a' such that $P(a',a') > 0$, and let $K = 2P(a',a')$, then for any $b\varepsilon S$,

$$\begin{aligned} K = 2P(a',a') &= 2P(a'eb,a'eb) \\ &= 2Q(e,e)[P(a',b) + P(b,a')] \\ &= P(a',b) + P(b,a'). \end{aligned}$$

Now, take any $a,\ b\varepsilon S$,

$$\begin{aligned} 2P(b,b) &= 2P(bea',bea') \\ &= P(b,a') + P(a',b) \\ &= K \\ &= 2P(bea,bea) \\ &= P(b,a) + P(a,b), \end{aligned}$$

so P is additive.

<u>Corollary</u>. <u>The core of a decomposable linear structure is a linear structure</u>.

Proof. Let P be the decomposable linear structure and Q its core. Choose $a^*, b^* \in S$ such that $P(a^*,b^*) > P(b^*,a^*)$; this is possible by D3. For any $\alpha, \beta \in E$, the fact that P is linear means

$$P(a^*\alpha b^*,c) - P(a^*\beta b^*,c) \quad \text{and} \quad P(c,a^*\beta b^*) - P(c,a^*\alpha b^*)$$

are both non-negative for all $c \in S$ or both are non-positive for all $c \in S$. Without loss of generality, suppose the former is true. Choose $c = a^*\lambda b^*$, where $\lambda \in E$, apply the decomposition assumption, and collect terms:

$$P(a^*,b^*)[Q(\alpha,\lambda) - Q(\beta,\lambda)] - P(b^*,a^*)[Q(\lambda,\beta) - Q(\lambda,\alpha)] \geq 0,$$

$$P(a^*,b^*)[Q(\lambda,\beta) - Q(\lambda,\alpha)] - P(b^*,a^*)[Q(\alpha,\lambda) - Q(\beta,\lambda)] \geq 0.$$

If P is reflexive and transitive, then $P(b^*,a^*) = 0$ and $P(a^*,b^*) > 0$, so

$$Q(\alpha,\lambda) \geq Q(\beta,\lambda) \quad \text{and} \quad Q(\lambda,\beta) \geq Q(\lambda,\alpha), \text{ for all } \lambda \in E$$

so Q is linear. If P is additive, theorems 5 and 6 state that Q is additive with $K = 1$, so

$$\begin{aligned} Q(\alpha,\lambda) - Q(\beta,\lambda) &= 1 - Q(\lambda,\alpha) - 1 + Q(\lambda,\beta) \\ &= Q(\lambda,\beta) - Q(\lambda,\alpha). \end{aligned}$$

Substituting,

$$[P(a^*,b^*) - P(b^*,a^*)][Q(\alpha,\lambda) - Q(\beta,\lambda)] \geq 0.$$

Since, by the choice of a^* and b^*, the first term is positive, the second must be non-negative, so Q is linear. By theorem 6, all cases have been covered.

While this theorem gives several necessary conditions for a discrimination structure to be decomposable, no necessary and sufficient conditions are now known. In section 9, a somewhat special sufficient condition will be presented.

It is interesting to note the relationship between this theorem and experimental practice. The additive case with $K = 1$ corresponds to forced choice responses, i.e., where a subject is not permitted to report indifference between two alternatives and $P(a,a)$ is taken to be 1/2 by definition. The alternative procedure is to permit indifference reports. One might be inclindd to postulate that a person will always report indifference between an element and itself, i.e., he will yield a reflexive discrimination structure. But in that case, the structure must be transitive: if $a > b$, then the subject must either report preference for a over b or indifference between them, but never preference for b over a. It is clear that one would have to be quite optimistic to expect this, and experimental practice in psychophysics, where an analogous problem exists, is to use forced choice questions.

5. Decomposable Discrimination Structures on Risk Spaces

To proceed further, it appears to be necessary to make somewhat stronger assumptions about the underlying space of alternatives, specifically to introduce the concept of subjective probabilities. The intuitive idea of subjective probability is reasonably clear, and a number of authors have used it to construct theories of decision making, the most elaborate being Savage's [7]. These authors have not, however, attempted to use the notion of imperfect discrimination to get at it, and, as I shall attempt to show, this approach seems to have certain empirical advantages. If E is a Boolean algebra of events, then intuitively a subjective probability function ϕ on E is a single valued function into the closed interval from 0 to 1 with the properties:

$$\phi(o) = 0$$

$$\phi(\alpha) + \phi(\overline{\alpha}) = 1.$$

The first of these is certainly not controversial, since in essence this is what is meant by the event o; this is not to say that it will be a trivial matter to identify o in applications. The second condition -- additivity -- has been objected to by at least one person: Edwards [1]. He has contended that, at least when the events in E have objective probabilities attached to them, subjective probability should be a single valued function of these probabilities, not just of the events. Thus, if α has objective probability 1/2, then so does $\overline{\alpha}$, hence by the additivity condition the subjective probability of the objective probability 1/2 must itself be 1/2. This he and others find objectionable for empirical reasons. He feels that the additivity condition must go. Others feel the difficulty lies in identifying objective probabilities, rather than events, as basic. I would hold that the subjective probabilities of heads and tails may and probably do differ, even though they have the same objective probability, but their sum must be 1.

On the face of it, it would appear that ϕ could not be determined; however, certain adaptations of tradition psychophysical methods seem to make it possible to find ϕ from data provided that the decomposition assumption is met (see section 7). To develop these methods, one very important condition has to be imposed upon E, namely: it must be so dense that the image of ϕ is the whole of the unit interval. I shall introduce these ideas axiomatically in definition 12, now using the symbols α and β to stand for the subjective probabilities attached to events, not for the events themselves.

Let $[\sigma,\rho]$ denote the closed interval $\{\alpha | \alpha$ real and $\sigma \leq \alpha \leq \rho\}$, (σ,ρ) the corresponding open interval and $(\sigma,\rho]$ and $[\sigma,\rho)$ the half open intervals.

<u>Definition 12</u>. A set S is called a <u>risk space</u> if for all $a,b \in S$ and $\alpha \in [0,1]$,

R1. $a\alpha b \in S$,

R2. $a\alpha a = a$,

R3. $a\alpha b = b(1-\alpha)a$,

R4. $a1b = a$,

R5. if either $\alpha \neq 0$ or $\beta \neq 0$,

$$a\alpha(b\beta c) = \left(a \frac{\alpha}{\alpha+\beta-\alpha\beta} b\right)(\alpha+\beta-\alpha\beta)c.$$

Given the two demands on a subjective probability function, R1-R4 follow immediately from the corresponding axioms in definition 7. Axiom R5 has no counterpart in definition 7, and it is controversial. It is almost certainly false if the probabilities α and β are taken to be objective, but at the subjective level it has a certain plausibility. Actually, no results in this paper, save theorem 14, require nearly so much as R5. Exactly what is used will be examined following the proof of theorem 7. What Hausner has called a <u>mixture space</u> is a risk space which also satisfies

R6. if $a\alpha c = b\alpha c$ for some $\alpha \varepsilon (0,1]$, then $a = b$.

(Note: R4 is then a consequence of R2, R5, and R6.) We will have no occasion to use this cancellation rule, and since it is controversial I have chosen to eliminate it from the concept of a risk space.

The next theorem, which will be discussed in detail following the proof and in succeeding sections, is one of the major results in this work. It should be emphasized that it holds for a risk space, not a weak mixture space.

<u>Theorem 7.</u> <u>Let</u> Q <u>be the core of a decomposable discrimination structure</u> P <u>over a risk space, then</u> $Q(\alpha,\beta) = Q(\alpha\beta)$. <u>Let</u> $R(\alpha) = Q(\alpha) - Q(-\alpha)$, <u>then</u> R <u>satisfies</u>

$$R(\alpha\beta) = R(\alpha)R(\beta), \quad R(\alpha) = -R(-\alpha), \text{ for } \alpha,\beta\varepsilon[-1,1],$$

$$R(\alpha)\varepsilon[-1,1], \quad R(0) = 0, \quad R(-1) = -1, \quad R(1) = 1.$$

<u>Let</u> $S(\alpha) = Q(\alpha) + Q(-\alpha)$. <u>If</u> P <u>is additive then</u> $S(\alpha) = 1$. <u>If</u> P <u>is reflexive, then</u>

$$S(\alpha\beta) = S(\alpha)S(\beta), \quad S(\alpha) = S(-\alpha), \text{ for } \alpha,\beta\varepsilon[-1,1],$$

$$S(\alpha)\varepsilon[0,1], \quad S(0) = 0, \quad S(1) = 1.$$

If Q *is continuous and* P *is additive, then there exists* $\varepsilon > 0$ *such that*

$$Q(\alpha) = \begin{cases} \dfrac{1 + \alpha^{\varepsilon}}{2}, & \text{if } \alpha \varepsilon [0,1] \\[2mm] \dfrac{1 - |\alpha|^{\varepsilon}}{2}, & \text{if } \alpha \varepsilon [-1,0]. \end{cases}$$

If Q *is continuous and* P *is reflexive, then there exist* $\varepsilon \geq \delta > 0$ *such that*

$$Q(\alpha) = \begin{cases} \dfrac{\alpha^{\delta} + \alpha^{\varepsilon}}{2}, & \text{if } \alpha \varepsilon [0,1] \\[2mm] \dfrac{|\alpha|^{\delta} - |\alpha|^{\varepsilon}}{2}, & \text{if } \alpha \varepsilon [-1,0] \text{ ;} \end{cases}$$

If P *is linear and reflexive, then* $Q(\alpha) = 0$ *for* $\alpha \leq 0$.

Proof. By axiom R5 of a risk space, $(a\alpha b)\beta b = a\alpha\beta b$, so

$$P[(a\lambda b)\alpha b, (a\lambda b)\beta b] = P(a\lambda\alpha b, a\lambda\beta b).$$

Using the decomposition assumption twice on the left and once on the right,

$$P(a\lambda b,b)Q(\alpha,\beta) + P(b,a\lambda b)Q(\beta,\alpha)$$
$$= P(a\lambda b,a0b)Q(\alpha,\beta) + P(a0b,a\lambda b)Q(\beta,\alpha)$$
$$= Q(\alpha,\beta)[P(a,b)Q(\lambda,0) + P(b,a)Q(0,\lambda)] + Q(\beta,\alpha)[P(a,b)Q(0,\lambda)+P(b,a)Q(\lambda,0)]$$
$$= P(a,b)Q(\alpha\lambda,\beta\lambda) + P(b,a)Q(\beta\lambda,\alpha\lambda)$$

Collecting terms,

$$P(a,b)[Q(\alpha,\beta)Q(\lambda,0) + Q(\beta,\alpha)Q(0,\lambda) - Q(\alpha\lambda,\beta\lambda)]$$
$$+ P(b,a)[Q(\alpha,\beta)Q(0,\lambda) + Q(\beta,\alpha)Q(\lambda,0) - Q(\beta\lambda,\alpha\lambda)] = 0.$$

Let a^*, b^* be the elements described in axiom D3. Consider the two equations obtained by the substitutions $a = a^*$, $b = b^*$ and $a = b^*$, $b = a^*$. Since by the choice of a^* and b^*, the determinant of coefficients $P(a^*,b^*)^2 - P(b^*,a^*)^2 \neq 0$,

$$Q(\alpha\lambda,\beta\lambda) = Q(\alpha,\beta)Q(\lambda,0) + Q(\beta,\alpha)Q(0,\lambda).$$

For any $\alpha \geq \beta$, $\beta/\alpha \leq 1$, so

$$Q(\alpha,\beta) = Q(\alpha, \frac{\beta}{\alpha}\alpha)$$
$$= Q(1,\beta/\alpha)Q(\alpha,0) + Q(\beta/\alpha,1)Q(0,\alpha).$$

Also,

$$Q(\alpha-\beta,0) = Q[\alpha(1-\frac{\beta}{\alpha}), \alpha.0]$$
$$= Q(1-\frac{\beta}{\alpha},0)Q(\alpha,0) + Q(0,1-\frac{\beta}{\alpha})Q(0,\alpha)$$

By theorem 5, Q is symmetric, so

$$Q(\alpha-\beta,0) = Q(1,\beta/\alpha)Q(\alpha,0) + Q(\beta/\alpha,1)Q(0,\alpha)$$
$$= Q(\alpha,\beta).$$

If $\alpha \leq \beta$, a similar result holds, so $Q(\alpha,\beta) = Q(\alpha-\beta)$.

Thus, the functional equation for Q can be written as:

$$Q(\alpha\beta) = Q(\alpha)Q(\beta) + Q(-\alpha)Q(-\beta).$$

We next show that $Q(-1) = 0$ and $Q(1) = 1$. From the functional equation,

$$Q(1) = Q(1)^2 + Q(-1)^2 \quad \text{and} \quad Q(-1) = 2Q(-1)Q(1).$$

From the second equation, either $Q(-1) = 0$ or $Q(1) = 1/2$. If $Q(1) = 1/2$, then the first equation yields $Q(-1)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$, so $Q(-1) = 1/2$. If $Q(-1) = 0$, then the first equation reduces to $Q(1) = Q(1)^2$, so $Q(1) = 0$ or 1. But in the proof of theorem 5 we showed $Q(1) \neq Q(-1)$, so $Q(1) = 1$ and $Q(-1) = 0$.

If R is defined as in the statement, then using the functional equation for Q,

$$R(\alpha\beta) = Q(\alpha\beta) - Q(-\alpha\beta)$$
$$= Q(\alpha)Q(\beta) + Q(-\alpha)Q(-\beta) - Q(-\alpha)Q(\beta) - Q(\alpha)Q(-\beta)$$
$$= [Q(\alpha) - Q(-\alpha)][Q(\beta) - Q(-\beta)]$$
$$= R(\alpha)R(\beta).$$

The other conditions on R follow immediately from the conditions on Q.

If S is defined as in the statement, then by theorems 5 and 6 $S \equiv 1$ for P additive. If P is reflexive, then

$$\begin{aligned} S(\alpha\beta) &= Q(\alpha\beta) + Q(-\alpha\beta) \\ &= Q(\alpha)Q(\beta) + Q(-\alpha)Q(-\beta) + Q(-\alpha)Q(\beta) + Q(\alpha)Q(-\beta) \\ &= [Q(\alpha) + Q(-\alpha)][Q(\beta) + Q(-\beta)] \\ &= S(\alpha)S(\beta). \end{aligned}$$

Again, the other conditions are obvious.

If Q is continuous, then so are R and S, and it is well known that the functional equation for R is solved by

$$R(\alpha) = \begin{cases} \alpha^{\varepsilon}, & \text{if } \alpha \varepsilon [0,1] \\ |\alpha|^{\varepsilon}, & \text{if } \alpha \varepsilon [-1,0], \end{cases}$$

for some $\varepsilon > 0$. Similarly, if P is reflexive,

$$S(\alpha) = |\alpha|^{\delta}, \quad \text{if } \alpha \varepsilon [-1,1]$$

for some $\delta > 0$. The expressions for Q are obtained by noting $Q = (R + S)/2$. Since $Q \geq 0$, it follows that $\varepsilon \geq \delta$.

Let P be linear and reflexive, and suppose $Q(-\lambda) > 0$ for $\lambda > 0$. Let $\alpha = (\lambda)^{1/2}$, then

$$Q(-\lambda) = Q(-\alpha.\alpha) = 2Q(\alpha)Q(-\alpha) > 0,$$

and so $Q(\alpha) > 0$ and $Q(-\alpha) > 0$. Thus,

$$Q(-\alpha,0) = Q(-\alpha) > 0 = Q(0) = Q(0,0),$$

and

$$Q(-\alpha,-\alpha) = Q(0) = 0 < Q(\alpha) = Q(0,\alpha),$$

hence Q is not linear. But this contradicts the corollary to theorem 6, so $Q(\alpha) = 0$ for $\alpha \leq 0$.

In addition to the continuous solutions to the core equations, two discontinuous ones are worthy of note. First,

$$Q(\alpha) = \begin{cases} 1\ , & \text{if } \alpha \varepsilon (0,1] \\ 0\ , & \text{if } \alpha = 0 \text{ and P is reflexive} \\ 1/2, & \text{if } \alpha = 0 \text{ and P is additive} \\ 0\ , & \text{if } \alpha \varepsilon [-1,0) \end{cases}$$

This represents perfect probability discrimination. Second, if P is additive,

$$Q(\alpha) = 1/2, \quad \text{if } \alpha \varepsilon (-1,1)$$

and if P is reflexive,

$$Q(\alpha) = \quad 0, \quad \text{if } \alpha \varepsilon [-1,1).$$

This core represents a total failure to discriminate between any probability pairs except the extremes 0 and 1. Later, in theorem 12, we will show that these are the only discontinuous cores for decomposable structures having linear utility functions.

Theorem 7, while not particularly difficult to prove, seems to be sufficiently important in its consequences to bear some scrutiny. First, it should be kept in mind that it holds for all decomposable discrimination structures over risk spaces, not just linear ones. Second, it rests on at most three assumptions which can be considered controversial, the degree to which they are controversial depending upon how the probabilities in the risk space are interpreted. The assumptions are axioms R3 and R5 and decomposability, i.e., statistical independence of preference and probability discrimination.

As indicated earlier, we shall come to interpret the probabilities as subjective probabilities attached to events -- the main reason, beside intuition, is given in section 7. However, a number of people have been anxious to use objective probabilities in utility theory, at least when the theory is interpreted normatively, so let us examine the three assumptions assuming that objective probabilities are intended.

Clearly, axiom R3 must be met, so we may turn to axiom R5. In the proof nothing so strong as axiom R5 is ever used; the assumption actually made is that

$$P[(a\lambda b)\alpha b,(a\lambda b)\beta b] = P(a\alpha\lambda b,a\beta\lambda b).$$

This would follow, for example, from the postulate $(a\alpha b)\beta b = a\alpha\beta b$, which can be paraphrased as saying that there is "no love or hate of gambling." Critics of modern utility theory have pointed out that, at least for objective probabilities, this assumption is more than likely in error; yet, it must be assumed if preferences are to be represented by linear utility functions, which are so important in most applications of utility theory to decision theory.

The independence assumption, it is true, has no tradition in utility theory, but it certainly has an honorable history in probability theory and its applications in science. In any case, if for whatever reasons these two assumptions are accepted, and if Q is assumed to be continuous as is customary in sensory psychology and as will be defended in theorem 12, then the form of Q is completely determined up to a single constant in the additive case and two constants in the reflexive case (which reduce to one if the structure is linear).

It is an empirical question of some interest to determine whether in fact these are the forms of the psychophysical functions for the discrimination of objective probabilities. If they are, then presumably the constants ε and δ are fairly basic parameters of human discrimination, and their distribution in the population should be investigated. As we shall see in section 10, it is of interest to know whether they are (approximately) constant or whether they vary from person to person.

It strikes me as doubtful that this experimental step can be bypassed, for the derived forms for Q do not seem completely outrageous in the light of one's intuition about probability discrimination and from what has been found for other psychophysical dimensions. Several cases are plotted in Figs. 1 and 2. One thing to note is that if there are any small errors, then with only slight smaller probability there are some quite large errors of judgment. Possibly, this corresponds to the sense of difficulty one has in making relative probability judgments except for the most extreme values.

If, however, we are not so fortunate -- and I do not believe we are -- as to have derived the correct form of the psychophysical function for objective probabilities, then to have a descriptive theory of preferences based upon objective probabilities one or the other or both of the major assumptions must go. This is to say, in at least one of two ways people must be inconsistent in their calculations of compound preferences from the simpler ones and from objective probabilities. Naive observation strongly suggests that R5 -- no love or hate of gambling -- should be dropped, but that would mean dropping linear utility functions which, in turn, would collapse most of decision theory. Thus, clearly, the most likely outcome is either a

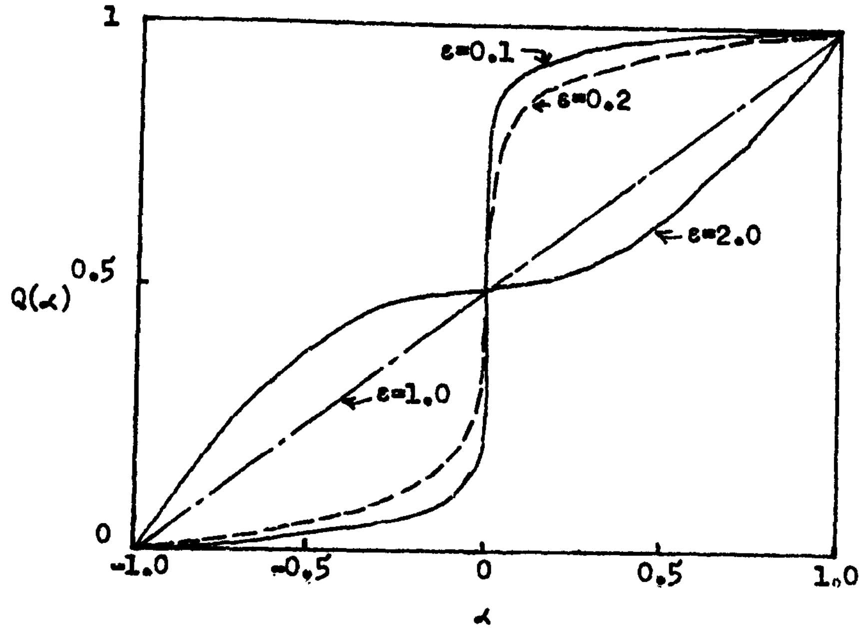

Fig. 1. Continuous cores for decomposable additive discrimination structures.

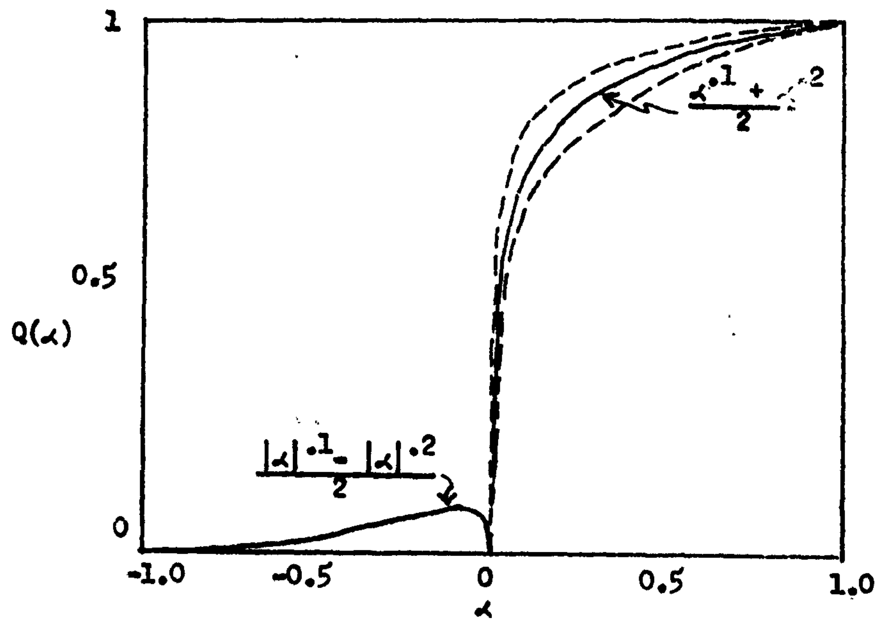

Fig. 2. A continuous core for decomposable reflexive discrimination structures.

hasty retreat from the independence assum ption or the admission that descriptive theories must be based upon some concept of subjective probability in the risk spaces. I shall argue for the latter course. Incidently, if it is accepted, it is not an unverifiable hypothesis which manages to save every conceivable empirical situation. The very specific conclusions of theorem 7 continue to place strong demands upon empirical data, even when subjective probability functions are admitted.

In connection with these points, a remark about normative interpretations is in order. It has been customary to argue that, while many of the assumptions of utility and decision theories are somewhat unrealistic for objective probabilities, they do represent a desirable form of consistent behavior; and, therefore, the theories can be treated as normative prescriptions for simplifying decision making. Certainly, R5 has been considered in this light, and it should not be too difficult to gain (from those ignorant of theorem 7) similar acceptance for the decomposability assumption; however, the theorem states that these normative considerations impose tight constraints on a psychophysical function. It is not at all clear that such functions are ever under conscious control or are even subject to serious modification by training. Thus, unless the derived functions are the actual forms for probability discrimination data, or unless we demand complete knowledge of and perfect discrimination among objective probabilities, this result raises some doubts about casually giving normative interpretations to inadequate descriptive theories.

6. Utility Functions and Sensation Scales

The following definition is in close conformity to traditional usage.

Definition 13. If P is a linear structure on S and the induced weak order is $\succsim$, then any real valued function on S which preserves the order of $\succsim$, i.e., $u(a) \geq u(b)$ if and only if $a \succsim b$, is called a utility function of P. If S is a risk space, a utility function u is called linear if

$$u(a\alpha b) = \alpha u(a) + (1-\alpha)u(b),$$

for all $a,b \varepsilon S$ and $\alpha \varepsilon [0,1]$.

If a linear structure P on S has the property that there exists $a^* \varepsilon S$ such that $a > b$ implies $P(a,a^*) > P(b,a^*)$, then $P(a,a^*)$ is a utility function of P. Many empirically interesting cases will probably have such an element a^*.

It is clear that if a linear structure has a utility function u, then any strictly increasing function of u is equally a utility function. Thus, if no further specifications are imposed, there is little if any point in introducing numerical representations of the weak orders. Historically, two quite distinct traditions for restricting the class of admissible utility function have developed. Utility theorists, assuming weak orders over mixture spaces, not linear structures over risk spaces, have concerned themselves with the existence of linear utility functions. The reasons are largely pragmatic: it has been practically impossible to devise mathematically interesting decision theories if utility expected values do not represent the utility of risky situations. The von Neumann axiom system was devised to give an intuitive justification for restricting one's attention to linear utility functions; the criticisms of it are well known and need not be entered into here.

At the same time, a much older tradition for selecting "utility" functions exists in psychophysics.[2] It traces back to the middle of the last century when Fechner postulated the equality of just noticeable differences as the defining property of subjective sensation. In more recent work, one postulates a linear structure P on the positive real line, and the idea — which we will specify more precisely in definition 14 — is to find those utility functions, if any, which render P dependent only upon utility differences. It is argued, largely on philosophical grounds, that a subjective sensation scale must have the property that the probability of detecting a difference on that scale depends only upon the difference and not upon its location on the scale. In practice, this definition has been found to be related to other intuitive concepts and to have been both useful and stimulating in psychophysics.

When S is taken to be the positive reals and P is assumed to be strictly monotonic in its two variable, conditions are known for the existence of such scales (which are unique up to a linear transformation) and their analytic form has been given. The reader should be warned that the traditional mathematical formulation of this

[2] The word "utility" as it is being used here is nothing but a mathematical label; it should not be interpreted as imputing values into psychophysical judgments.

problem is in error; the correct formulation and solution may be found in [5]. The magnitude of the error is sufficiently large that it now appears that some incorrect conclusions probably have been drawn from empirical data.

I know of no attempt to interrelate these two distinct traditions. Presumably, this is because, on the one hand, there has not been a probabilistic theory of utility, so it has not been possible to impose the psychological condition there; and, on the other hand, the idea of a risk space seems to make no sense in traditional sensory psychology, so linearity could not be imposed there. However, once a general probability model is postulated, the psychological condition can be extended to all linear structures having utility functions, and so in the domain of risk spaces there are two, apparently conflicting, criteria of selection. Part of the material in this section (theorem 8) and part of that in section 9 (theorem 15) establish that for an interesting class of linear structures on risk spaces there is no conflict: the two concepts are the same.

Definition 14. Let P be a linear structure with a utility function u, u is said to be a sensation scale if for all $a,b \varepsilon S$, $P[u(a),u(b)] = P(a,b)$ depends only upon $u(a) - u(b)$.

Theorem 8. Let P be a linear structure on a risk space and suppose P has a linear utility function u, then u is a sensation scale if and only if P is regular.

Proof. Suppose P is regular. Let $a,b,c,d \varepsilon S$ be such that $u(a)-u(b) = u(c)-u(d)$. We must show $P(a,b) = P(c,d)$. Suppose, with no loss of generality, that $u(b) \leq u(d)$. We consider the case $u(a)-u(b) > 0$; the other case is similar. Thus, $u(b) < u(a) \leq u(c)$. By the linearity of u, there exists an $\alpha \varepsilon (0,1]$ such that

$$u(a) = \alpha u(b) + (1-\alpha)u(c) = u(b\alpha c),$$

so $a \sim b\alpha c$. Hence, $P(a,b) = P(b\alpha c,b)$. Using the assumed relation of the elements and the linearity of u,

$$\begin{aligned} u(d) &= u(c) - u(a) + u(b) \\ &= u(c) - u(b\alpha c) + u(b) \\ &= \alpha u(c) + (1-\alpha)u(b) \\ &= u(c\alpha b), \end{aligned}$$

so $d \sim c\alpha b$. Thus, $P(c,d) = P(c,c\alpha b)$. By the regularity of P, $P(c,c\alpha b) = P(b\alpha c,b)$, so $P(a,b) = P(c,d)$.

Conversely, suppose u is a sensation scale. By linearity,

$$\begin{aligned} u(a) - u(a\alpha b) &= (1-\alpha)[u(a)-u(b)] \\ &= u(b\alpha a) - u(b), \end{aligned}$$

for any $a,b\varepsilon S$ and $\alpha \varepsilon [0,1]$. Since u is a sensation scale,

$$P(a,a\alpha b) = P[u(a)-u(a\alpha b)] = P[u(b\alpha a)-u(b)] = P(b\alpha a,b),$$

and so P is regular.

<u>Corollary</u>. <u>Any linear utility function of a decomposable linear structure is a sensation scale</u>.

Proof. By theorem 6 a decomposable linear structure is regular, so by theorem 8 a linear utility function is a sensation scale.

This corollary means that any decomposable linear structure with a linear utility function u can be represented completely by the utility function and another real valued function of a real variable, namely $P[u(a) - u(b)]$.

We will return to this general topic in theorem 15, where it is shown under slightly stronger conditions that a sensation scale is a linear utility function.

The concept of regularity has its parallel in the semiorders induced by linear structures.

Definition 15. A semiorder (L,I) on a weak mixture space S is called *regular* if for every $a,b \in S$ and $\alpha \in E$, $aI(a\alpha b)$ implies $(b\alpha a)Ib$.

Theorem 9. *Let* P *be a regular linear structure on a weak mixture space and let* k *be a number,* $\frac{1}{2} \leq k < 1$, *then the semiorder* (L_k, I_k) *induced by definition 5 is regular.*

Proof. By definition 5, $aI_k(a\alpha b)$ is equivalent to $P(a,a\alpha b) \leq k$ and $P(a\alpha b,a) \leq k$. By the regularity of P, $P(b\alpha a,b) = P(a,a\alpha b) \leq k$ and $P(b,b\alpha a) = P(a\alpha b,a) \leq k$, so $(b\alpha a)I_k b$.

It is plausible that an analogue to theorem 8 holds for semiorders, with the notion of jnd functions (see [4]) replacing the probability of discrimination. This is true; however, an important modification is necessary: the assumed linear utility function must be unbounded, otherwise the obvious boundary effect prevents the jnd functions from being everywhere constant. (The following theorem was presented in technical report 13 and is repeated here for the sake of completeness.)

Theorem 10. *Let* (L,I) *be a semiorder on a risk space, and suppose it has a linear utility function*[3] *whose jnd functions are everywhere*

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[3] Definition 3 of [4] is intended here.

<u>different from zero. The jnd functions are constant and equal if and only if u is unbounded and the semiorder is regular.</u>

Proof. Suppose the jnd functions are constant and equal. Since they are different from zero, u must be unbounded. Suppose $aI(a\alpha b)$, then by theorem 2 of [4],

$$u(a) - \underline{\delta}(a) \leq u(a\alpha b) \leq u(a) + \overline{\delta}(a).$$

Using the linearity of u and subtracting u(a),

$$-\underline{\delta}(a) \leq (1-\alpha)[u(b)-u(a)] \leq \overline{\delta}(a).$$

From the equality and constancy of the jnd functions,

$$-\overline{\delta}(b) \leq (1-\alpha)[u(b)-u(a)] \leq \underline{\delta}(b).$$

Adding $-u(b)$, changing signs, and using the linearity of u, we obtain

$$u(b) + \overline{\delta}(b) \geq \alpha u(b) + (1-\alpha)u(a) = u(b\alpha a) \geq u(b) - \underline{\delta}(b).$$

So, by theorem 2 of [4], $(b\alpha a)Ib$, and the semiorder is regular.

Conversely, suppose that u is unbounded and that the semiorder is regular. It will be recalled [4] that for $a,b \varepsilon S$ such that aIb, the two quantities

$$\overline{\alpha}(a,b) = \sup[\alpha \mid (a\alpha b)Ib]$$
$$\underline{\alpha}(a,b) = \sup[\alpha \mid (b\alpha a)Ia],$$

were defined; these were shown to be closely related to linear utility functions. For regular semiorders, we show that they are equal. By the regularity of the semiorder,

$$\begin{aligned}\overline{\alpha}(a,b) &= \sup[\alpha \mid (a\alpha b)Ib] \\ &\leq \sup[\alpha \mid (b\alpha a)Ia] \\ &= \underline{\alpha}(a,b).\end{aligned}$$

Similarly, $\underline{\alpha}(a,b) \leq \overline{\alpha}(a,b)$, so they are equal

Consider any $a\varepsilon S$. Since u is unbounded, there exist $a^*, b^* \varepsilon S$ such that a^*La and aLb^*. With no loss of generality, we may choose u to be that linear function with $u(a^*) = 1$ and $u(b^*) = 0$. Since the jnd functions never equal zero, we know by the remarks following theorem 4 of [4] that

$$\bar{\delta}(a) = \bar{\alpha}(a^*, a)[1-u(a)] \quad \text{and} \quad \underline{\delta}(a) = \underline{\alpha}(a, b^*)u(a).$$

Since u is linear, there exists $\alpha \varepsilon_{[0,1]}$ such that $a \backsim a^* \alpha b^*$, where $\backsim$ denotes the equivalence relation induced by a semiorder, and $u(a) = \alpha$.

Thus,

$$\begin{aligned} \bar{\delta}(a) &= \bar{\alpha}(a^*, a^*\alpha b^*)[1-\alpha] & \text{and} \quad \underline{\delta}(a) &= \underline{\alpha}(a^*\alpha b^*, b^*)\alpha \\ &= \underline{\alpha}(a^*, a^*\alpha b^*)[1-\alpha] & &= \bar{\alpha}(a^*\alpha b^*, b^*)\alpha. \end{aligned}$$

We now show $\underline{\alpha}(a^*, a^*\alpha b^*) = \frac{1}{1-\alpha}\underline{\alpha}(a^*, b^*)$ and $\bar{\alpha}(a^*\alpha b^*, b^*) = \frac{1}{\alpha}\bar{\alpha}(a^*, b^*)$.

Observe, $a^*L(a^*\alpha b^*)$, so by definition,

$$\underline{\alpha}(a^*, a^*\alpha b^*) = \sup \{\beta \mid [(a^*\alpha b^*)\beta a^*]Ia^*\}.$$

by axioms R3 and R5 of a risk space,

$$\begin{aligned} (a^*\alpha b^*)\beta a^* &= [b^*(1-\alpha)a^*]\beta a^* \\ &= b^*(1-\alpha)\beta a^*. \end{aligned}$$

Thus,

$$\begin{aligned} \underline{\alpha}(a^*, a^*\alpha b^*) &= \sup \{\beta \mid [b^*(1-\alpha)\beta a^*]Ia^*\} \\ &= \frac{1}{1-\alpha}\sup[\beta \mid (b^*\beta a^*)Ia^*] \\ &= \frac{1}{1-\alpha}\underline{\alpha}(a^*, b^*), \end{aligned}$$

since a^*Lb^*. The other case is similar. Using these results and the fact that $\underline{\alpha}(a^*, b^*) = \bar{\alpha}(a^*, b^*)$, we have

$$\bar{\delta}(a) = \bar{\alpha}(a^*, b^*) = \underline{\delta}(a).$$

A simple induction now establishes that the jnd functions are constant and equal.

7. Subjective Probability

Recall that in theorem 7 it was established that the core Q of a decomposable discrimination structure on a risk space has the property that $Q(\alpha,\beta) = Q(\alpha-\beta)$ and that in the corollary to theorem 6 it was shown that the core of a decomposable linear structure is linear. Thus, for those linear structures where the weak order induced upon [0,1] by the core is the natural ordering of [0,1] by numerical magnitude, the underlying probabilities of the risk space form a sensation scale for Q. As is easily verified, this is the case when Q is continuous -- an assumption which, as we shall see in the next theorem, probably will always be made in empirical work and in applications of this theory. Thus, if there is any sense to the psychological assumption that the sensation scales of linear structures represent subjective sensation, then for linear structures on risk spaces we have shown that the assumptions of decomposability and of continuity of the core imply the probabilities entering into the risk space must be subjective probabilities. The implications of this for experimental practice and a sketch of how empirical data can be used to get the subjective probability function will be presented after the next result.

Since these conclusions seem to be important for utility theory, it is of interest to know whether the assumption of continuity of the core is a serious restriction or whether it is safe to assume that it is generally met. The following condition, which will play an important role in the next two sections, seems adequate. It will be noted that if a linear structure has a linear utility function the condition is satisfied.

Definition 16. A linear structure on a weak risk space is called consistent if for every $a, b \in S$ such that $a > b$ and $P(a,b) \neq P(b,a)$ and for every $\alpha, \beta \in [0,1]$ such that $\alpha > \beta$, then $a\alpha b \gtrsim a\beta b$.

Theorem 11. The core of a decomposable consistent linear structure is either continuous or one of the two discontinuous cores defined following theorem 7.

Proof. Let P be the structure and Q its core. We first show that if $\alpha \geq \beta \geq 0$, then $Q(\alpha) \geq Q(\beta)$. By axiom D3, there exist $a^*, b^* \in S$ such that $P(a^*,b^*) \neq P(b^*,a^*)$, and there is no loss of generality in assuming $a^* > b^*$. Consider $\alpha \geq \beta \geq 0$, then by definition 2 and the consistency assumption, $P(a^*\alpha b^*, b^*) \geq P(a^*\beta b^*, b^*)$. Using the decomposition assumption and theorem 7,

$$P(a^*\alpha b^*, b^*) = P(a^*,b^*)Q(\alpha) + P(b^*,a^*)Q(-\alpha)$$

$$P(a^*\beta b^*, b^*) = P(a^*,b^*)Q(\beta) + P(b^*,a^*)Q(-\beta).$$

By theorem 6, P is either transitive or additive. If it is transitive, $P(a^*,b^*) > P(b^*,a^*) = 0$, so $Q(\alpha) \geq Q(\beta)$. If it is additive, then by theorem 5, $Q(\alpha) + Q(-\alpha) = 1$, so

$$0 \leq P(a^*\alpha b^*, b^*) - P(a^*\beta b^*, b^*)$$

$$= [P(a^*,b^*) - P(b^*,a^*)][Q(\alpha) - Q(\beta)].$$

Since, by choice, the first term is positive, $Q(\alpha) \geq Q(\beta)$.

Let $R(\alpha) = Q(\alpha) - Q(-\alpha)$. If $\alpha \geq 0$, then by theorems 5, 6, and 7,

$$R(\alpha) = \begin{cases} 2Q(\alpha) - 1 & \text{, if } P \text{ is additive} \\ Q(\alpha) & \text{, if } P \text{ is reflexive} \end{cases},$$

If Q has a discontinuity at $\lambda \in (0,1)$, so does R. Since Q is non-decreasing, so is R, hence the discontinuity is a jump. For any α, $\lambda \leq \alpha \leq 1$, and $\beta = \lambda/\alpha$, theorem 7 implies

$$R(\lambda) = R(\alpha\beta) = R(\alpha)R(\beta),$$

so there is a discontinuity at either α or β. Since $\lambda < 1$ and R

is non-decreasing, there are a non-denumerable number of jumps, which is impossible since R is bounded. Thus, the only discontinuities can be at 0, 1, and -1.

Suppose 1 is a discontinuity, then there exists $\delta > 0$ such that for any $\alpha\varepsilon[0,1)$, $R(\alpha) \leq 1-\delta$. Consider any $\beta\varepsilon[0,1)$ and any integer $n > 0$. Clearly, there exists an $\alpha\varepsilon[0,1)$ such that $\alpha^n = \beta$, so by induction

$$R(\beta) = R(\alpha^n) = R(\alpha)^n \leq (1-\delta)^n,$$

so $R(\alpha) = 0$ for $\alpha\varepsilon(-1,1)$. If P is transitive, then $Q(\alpha) = 0$, and if P is additive, then $Q(\alpha) = 1/2$, for $\alpha\varepsilon(-1,1)$. This is the second discontinuous solution described after theorem 7.

Next, suppose there is a discontinuity at 0. Suppose P, and therefore Q, is transitive and that there is a $\lambda\varepsilon(0,1)$ such that $Q(\lambda) = 1 - \delta < 1$. For any $\varepsilon > 0$, there exists an integer n such that $Q(\lambda^n) = Q(\lambda)^n = (1-\delta)^n < \varepsilon$. Since Q is non-decreasing and $Q(0) = 0$, this shows Q is continuous at 0, contrary to assumption. If Q is additive, a parallel argument holds using R. Thus, $Q(\alpha) = 1$ for $\alpha\varepsilon(0,1]$, which is the first solution described, and the proof is concluded.

These two discontinuous solutions represent, it will be recalled, total failure to discriminate probability differences except for the most extreme values and perfect discrimination, neither of which people can be expected to exhibit. Thus, all that can arise in practice under our assumptions are the continuous solutions. This means that if the decomposability assumption is met by a consistent linear structure, and so by any linear structure having a linear utility function, then the risk space must be constructed from subjective probabilities, where a subjective probability function is defined to be the sensation scale of the linear structure characterizing the discrimination of the underlying event space.

With this interpretation of the probabilities of a risk space, it is no longer apparent that axiom R5 is false; however, axiom R3 is now also thrown into doubt. These and the decomposition assumption will have to be examined empirically. A plausible empirical procedure is outlined below.

A basic set of independent events is chosen (e.g., including coin tossing, the next day's weather, the fifth letter of a certain line of a certain page of a certain book, etc.) from which E is constructed, and the probabilities of preference among pure and risky alternatives based upon these events are discovered. Two cases should be examined: monetary gambles and non-monetary ones. As we shall see, theorem 13 leads one to suspect that there may be quite striking differences between these two cases. From these data, the right side of the formula following definition 8 is computed. If it is not independent of the alternatives a and b, then the decomposition assumption is not valid, so none of the results apply and we are stuck. If, however, it is independent of the alternatives, then we have an empirical estimate of Q. From Q, the ordering $\gtrsim$ is determined according to definition 2; if this is not a weak order, again we are stuck. If, however, it is, then choose any of its utility functions, call it u, such that $u(e) = 1$ and $u(o) = 0$. We can then transform the data for the discrimination of events into discrimination of the real numbers $u(\alpha)$ by defining

$$Q[u(\alpha),u(\beta)] = Q(\alpha,\beta), \quad \text{where } \alpha,\beta\varepsilon E.$$

This now formulates a conventional psychophysical problem, and the standard calculations, described to some extent in [5], can be adapted to find an approximation to the function π which transforms u into a sensation scale. The function ϕ, where $\phi(\alpha) = \pi[u(\alpha)]$, is then the subjective probability function for the event space E. With ϕ known, the additivity postulate $\phi(\alpha) = 1-\phi(\overline{\alpha})$ can be checked.

If the assumptions underlying theorem 7 are valid, then we know that the empirical data Q and the function ϕ must be related as follows: in the additive case,

$$Q(\alpha,\beta) = \begin{cases} \dfrac{1 + [\phi(\alpha) - \phi(\beta)]^{\varepsilon}}{2}, & \text{if } \phi(\alpha) \geq \phi(\beta) \\ \dfrac{1 - [\phi(\beta) - \phi(\alpha)]^{\varepsilon}}{2}, & \text{if } \phi(\alpha) \leq \phi(\beta) \end{cases}$$

and in the transitive case,

$$Q(\alpha,\beta) = \begin{cases} [\phi(\alpha) - \phi(\beta)]^{\varepsilon}, & \text{if } \phi(\alpha) \geq \phi(\beta) \\ 0, & \text{if } \phi(\alpha) \leq \phi(\beta) \end{cases}$$

where $\alpha,\beta\varepsilon E$ and $\varepsilon > 0$. With ϕ known, it is a matter of estimation -- probably by curve fitting -- to obtain ε.

It will be noted that each of the central assumptions is tested separately. The decomposition assumption first, then the linearity of P (either directly or via the linearity of Q), R3 via the additivity of ϕ once it is computed, and finally R5 by finding whether Q has the correct form or not.

It is to be hoped that some experimentalist will find this (very likely exacting) task of interest. It is certainly intriguing to find out whether the discrimination of preference and of subjective probability are statistically independent processes, and, if so, whether subjective probabilities meet the additivity condition.

Furthermore, if E is chosen to consist of events having well defined objective probabilities, there will be interest in the relation between subjective and objective probabilities -- a topic which has received some speculative attention in the past.

8. Archimedean Structures

In this and the following section we shall be concerned with a particular class of linear structures which, as we shall see in theorem 14, have linear utility functions provided that there is neither perfect discrimination nor total failure to discriminate among the probabilities of the underlying risk space. The new restriction is one of the axioms imposed by von Neumann on weak orders to prove the existence of linear utility functions in that context.

Definition 17. A linear structure on a weak risk space S with the induced weak order $\gtrsim$ is said to be an Archimedean structure if, for all $a,b,c \in S$ satisfying $a > b > c$, there exists an $\alpha \in (0,1)$, such that $b \sim a\alpha c$.

Clearly, any linear structure having a linear utility function is Archimedean.

The next result, while mildly interesting itself, is primarily needed to prove the following two more important theorems.

Theorem 12. A decomposable Archimedean structure on a risk space is consistent.

Proof. Consider $a,b \in S$ such that $a > b$ and $P(a,b) \neq P(b,a)$ and $\alpha,\beta \in [0,1]$ such that $\alpha \geq \beta$. First, we show that $a\alpha b \gtrsim a\beta b$ if the core has the property that $Q(\sigma) = Q(-\sigma)$ for all $\sigma \in [0,1]$. Consider any $c \in S$, then if $c > a > b$, the Archimedean condition states that there exists $\lambda\varepsilon(0,1)$ such that $a \sim c\lambda b$. Using axiom R4 and the decomposition assumption,

$$\begin{aligned} P(a,b) &= P(c\lambda b, c0b) \\ &= P(c,b)Q(\lambda) + P(b,c)Q(-\lambda) \\ &= Q(\lambda)[P(c,b) + P(b,c)], \end{aligned}$$

and

$$\begin{aligned} P(b,a) &= P(c0b, c\lambda b) \\ &= Q(\lambda)[P(c,b) + P(b,c)]. \end{aligned}$$

Thus, $P(a,b) = P(b,a)$, which is contrary to choice, so $a \gtrsim c$. Similarly, $c \gtrsim b$.

Using either the Archimedean condition or axioms R3 and R4, for any $c\varepsilon S$ there exists $\sigma\varepsilon[0,1]$ such that $c \sim a\sigma b$. We distinguish three cases:

i. $\alpha = 0$. Then $\beta = 0$, so $a\alpha b = b = a\beta b$.

ii. $\alpha = 1$ and $\sigma = 0$. In this case, $c \sim b$, so

$$\begin{aligned} P(a\beta b,c) &= P(a\beta b,b) \\ &= P(a,b)Q(\beta) + P(b,a)Q(-\beta) \\ &\leq P(a,b)[Q(\beta) + Q(-\beta)] \\ &\leq P(a,b) \\ &= P(a1b,b) \end{aligned}$$

iii. All other values of α and σ. Note, $\alpha-\sigma\varepsilon(-1,1)$, so

$$\begin{aligned} P(a\alpha b,c) &= P(a\alpha b,a\sigma b) \\ &= P(a,b)Q(\alpha-\sigma) + P(b,a)Q(\sigma-\alpha) \\ &= Q(\alpha-\sigma)[P(a,b) + P(b,a)]. \end{aligned}$$

In a similar way,

$$P(a\beta b,c) = Q(\beta-\sigma)[P(a,b) + P(b,a)].$$

We now show that $Q(\rho) = Q(\lambda)$ if $\rho,\lambda\varepsilon(-1,1)$. If P is additive, then by theorem 5, $1 = Q(\rho) + Q(-\rho) = 2Q(\rho)$, for $\rho\varepsilon(-1,1)$, so $Q(\rho) = 1/2$. If P is reflexive and transitive, then since $a\rho b \gtrsim b$ and $P(b,a) = 0$,

$$0 = P(a0b,a\rho b) = Q(-\rho)P(a,b),$$

so $Q(\rho) = Q(-\rho) = 0$. With this equality, we may conclude

$$P(a\alpha b,c) = P(a\beta b,c).$$

Combining these cases, $P(a\alpha b,c) \geq P(a\beta b,c)$, for all $c\varepsilon S$. Similarly, $P(c,a\alpha b) \leq P(c,a\beta b)$, for all $c\varepsilon S$, so by definition 2, $a\alpha b \gtrsim a\beta b$.

Now, suppose that $a\beta b > a\alpha b$, then clearly $\alpha > \beta$ and by what we have just shown there exists $\lambda_o\varepsilon[0,1)$ such that $Q(\lambda_o) \neq Q(-\lambda_o)$. First, we show that $a > a\beta b$. If not, then by the linearity of P, $a\beta b \gtrsim a > b$. By the Archimedean condition or axiom R4 there exists $\rho\varepsilon(0,1]$ such that $a \backsim (a\beta b)\rho b$. By axiom R5, $a \backsim a\beta\rho b$. By the decomposition assumption and axiom R4,

$$\begin{aligned} P(a,a) &= P(a\beta\rho b,a1b) \\ &= P(a,b)Q(\beta\rho-1) + P(b,a)Q(1-\beta\rho) \\ &= P(a1b,a\beta\rho b) \\ &= P(a,b)Q(1-\beta\rho) + P(b,a)Q(\beta\rho-1). \end{aligned}$$

Subtracting,

$$0 = [P(a,b) - P(b,a)][Q(\beta\rho-1) - Q(1-\beta\rho)].$$

By choice, the first term is non-zero, so $Q(\beta\rho-1) = Q(1-\beta\rho)$.

If $\sigma_o\varepsilon[0,1]$ has the property $Q(\sigma_o) = Q(-\sigma_o)$, then we show $Q(\sigma) = Q(-\sigma)$ for all $\sigma\varepsilon[0,\sigma_o]$. Let $\sigma = \lambda\sigma_o$, where $\lambda\varepsilon[0,1]$, then by the functional equation derived in theorem 7,

$$Q(\sigma) = Q(\lambda\sigma_o) = Q(\lambda)Q(\sigma_o) + Q(-\lambda)Q(-\sigma_o)$$

$$= Q(\sigma_o)[Q(\lambda) + Q(-\lambda)],$$

and

$$Q(-\sigma) = Q(-\lambda\sigma_o) = Q(-\lambda)Q(\sigma_o) + Q(\lambda)Q(-\sigma_o)$$
$$= Q(\sigma_o)[Q(\lambda) + Q(-\lambda)].$$

Thus, $Q(\sigma) = Q(-\sigma)$. By hypothesis, there exists $\lambda_o \varepsilon [0,1)$ such that $Q(\lambda_o) \neq Q(-\lambda_o)$, so for any $\lambda \geq \lambda_o$, the inequality holds. If $\sigma_o > 0$, then clearly σ_o and λ_o can be chosen arbitrarily close, so there exists $\sigma > \lambda_o$ such that $\sigma^2 < \sigma_o$. Using the functional equation for Q,

$$Q(\sigma^2) = Q(\sigma)^2 + Q(-\sigma)^2$$
$$= Q(-\sigma^2)$$
$$= 2Q(\sigma)Q(-\sigma),$$

so $[Q(\sigma) - Q(-\sigma)]^2 = 0$, which is contrary to the choice of σ. Thus, $\sigma_o = 0$, so $\beta\sigma = 1$. But this is impossible since $\rho \leq 1$ and $\beta < \alpha \leq 1$, so we must conclude $a > a\beta b > a\alpha b$.

The Archimedean condition is used once again to find $\lambda \varepsilon (0,1)$ such that $a\beta b \sim a\lambda(a\alpha b) = a(\alpha+\lambda-\alpha\lambda)b$. An argument similar to that just employed establishes that $\beta = \alpha + (1-\alpha)\lambda$, which is impossible since $\beta < \alpha$. Thus, P is consistent.

<u>Theorem 13</u>. <u>Let</u> P <u>be a decomposable Archimedean structure on a risk space</u> S <u>and let</u> Q <u>be its core</u>. <u>If there are elements</u> $a,b \varepsilon S$ <u>such that</u> $P(b,a) = 0$ <u>and either</u> $P(a,b) = K$ <u>in the additive case or</u> $P(a,b) = 1$ <u>in the transitive case, then either</u> $Q(\alpha) = 1$ <u>for</u> $\alpha \varepsilon (0,1]$ <u>or</u> $a \gtrsim c \gtrsim b$ <u>for all</u> $c \varepsilon S$.

Proof. Let C denote K in the additive case and 1 in the transitive case. Suppose $c > a$, then by definition 2, $P(c,b) \geq P(a,b) = C$, so $P(c,b) = C$

and $P(b,c) = 0$ by definitions 1 and 10. By the Archimedean condition, there exists an $\alpha \varepsilon (0,1)$, such that $a \sim c\alpha b$, so

$$P(a,b) = C = P(c\alpha b, c0b)$$
$$= P(c,b)Q(\alpha) + P(b,c)Q(\sim\alpha)$$
$$= CQ(\alpha).$$

Thus, $Q(\alpha) = 1$ for an $\alpha \varepsilon (0,1)$. By theorem 12, P is consistent, so by theorem 11, $Q(\alpha) = 1$ for all $\alpha \varepsilon (0,1]$. Similarly, either $c \gtrsim b$ or the same core exists.

This result seems to have considerable import for experiments and applications of utility theory. Any realization of a risk space is generated by forming probability "mixtures" from a finite number of "pure" or "basic" alternatives. It is probably safe to assume that usually there will be more than two alternatives and that the subjects will be incapable of perfect discrimination among the underlying events (i.e., subjective probabilities) of the risk space (i.e., not $Q(\alpha) = 1$ for all $\alpha \varepsilon (0,1]$). Now, if the subject is capable of perfect discrimination among three or more basic alternatives, then the theorem establishes that either his preferences cannot be represented by a linear utility function or his preference discrimination between alternatives is not independent of his discrimination between (subjective) probabilities. Judging by our almost daily fluctuations in preferences among many classes of alternatives, this result is probably not of very serious consequence for a wide class of situations; however, such fluctuation may not occur for one very important commodity — money. It seems plausible to assume that, other things being equal, subjects will invariably prefer a larger to a smaller sum.[4] If such is the case and if there is any reason to believe that the utility for a gamble of money is given by the expected value of the component utilities, then the preference structure for gambles cannot arise from independent judgments of preference among the alternatives and judgments of the magnitudes of the probabilities involved. Preferences must statistically influence the perception of probability differences! If such distortion does not occur, then the utility for money cannot be a linear utility function. In any event, since most, if not all, of the experimental studies relating to utility theory have involved money gambling situations, considerable caution must be exercised in generalizing such experimental results to commodities and alternatives which differ from money in not possessing a culturally accepted simple ordering.

9. Existence of Linear Utility Functions for Decomposable Archimedean Structures

The next result establishes that all decomposable Archimedean structures on a risk space, except those having one or the other of the possible discontinuous cores, have linear utility functions. It will be noted that this existence theorem employs somewhat different assumptions from those used by von Neumann. The two systems have R1-R5 and the Archimedean condition in common. The decomposability and continuity assumptions are different; indeed, they have no meaning for weakly ordered sets. The von Neumann axioms not assumed are:

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[4] In discussing this result with Ward Edwards, he pointed out that some of his subjects have exhibited peculiar reversals, e.g., sometimes preferring $4.00 to $4.03. However, if the set of alternatives included $1, 2, and 3, it is doubtful that such reversals ever will be exhibited for any of the three pairs, in which case our conclusion holds.

i. if $a \gtrsim b$ and $\alpha\varepsilon(0,1)$, then $a\alpha c \gtrsim b\alpha c$ for any $c\varepsilon S$,

ii. if $a\alpha c \gtrsim b\alpha c$ for some $\alpha\varepsilon(0,1)$, then $a \gtrsim b$,

and iii. axiom R6.

We could prove existence by showing that the Archimedean and decomposability assumptions (the latter in the form of consistency) imply i, ii and iii; however, since we can get a more precise result than mere existence, we will go about it directly.

Theorem 14. Let P be a discrimination structure on a risk space S. P is a decomposable Archimedean structure with continuous core Q if and only if there is a linear utility function u and P is of the following form: let $a^*, b^* \varepsilon S$ be elements described in axiom D3 with $a^* > b^*$ and let u be chosen so that $u(a^*) = 1$ and $u(b^*) = 0$, then

i. if P is additive, there is an $\varepsilon > 0$ such that

$$P(a,b) = \begin{cases} \frac{1}{2}\left\{K + [P(a^*,b^*)-P(b^*,a^*)][u(a)-u(b)]^{\varepsilon}\right\} & \text{, if } u(a) \geq u(b) \\ \frac{1}{2}\left\{K - [P(a^*,b^*)-P(b^*,a^*)][u(b)-u(a)]^{\varepsilon}\right\} & \text{, if } u(a) \leq u(b) \end{cases}$$

and ii. if P is reflexive, there is an $\varepsilon > 0$ such that

$$P(a,b) = \begin{cases} P(a^*,b^*)[u(a)-u(b)]^{\varepsilon} & \text{, if } u(a) > u(b) \\ 0 & \text{, if } u(a) < u(b) \end{cases} .$$

In both cases,

$$u(a) = \begin{cases} \left[\dfrac{P(a,b^*) - P(b^*,a)}{P(a^*,b^*)-P(b^*,a^*)}\right]^{1/\varepsilon} & \text{, if } a > b^* \\ 1 - \left[\dfrac{P(a^*,a)-P(a,a^*)}{P(a^*,b^*)-P(b^*,a^*)}\right]^{1/\varepsilon} & \text{, if } a < b^* \end{cases} .$$

Proof. Suppose that P is decomposable with a continuous core and that it is Archimedean. By theorem 7, we know the form of the core up to a constant ε; define u as in the statement of this theorem using this ε. Clearly $u(a^*) = 1$ and $u(b^*) = 0$. Consider any other $a \varepsilon S$, and there are three cases to consider:

i. If $a > a^* > b^*$, then by the Archimedean condition there exists an $\alpha \varepsilon (0,1)$ such that $a^* \backsim a\alpha b^*$. By the decomposability assumption and the axioms of a risk space,

$$P(a^*,b^*) = P(a\alpha b^*, a0b^*)$$
$$= P(a,b^*)Q(\alpha) + P(b^*,a)Q(-\alpha)$$

and

$$P(b^*,a^*) = P(a0b^*, a\alpha b^*)$$
$$= P(a,b^*)Q(-\alpha) + P(b^*,a)Q(\alpha).$$

Subtracting and substituting the known form for Q,

$$P(a^*,b^*)-P(b^*,a^*) = [P(a,b^*)-P(b^*,a)]\alpha^{\varepsilon},$$

so $u(a) = 1/\alpha$.

ii. If $a^* > a > b^*$, then there exists an α such that $a \backsim a^*\alpha b^*$, and by a similar argument, $u(a) = \alpha$.

iii. If $a^* > b^* > a$, then there exists an α such that $b^* \backsim a^*\alpha a$, and by a similar argument, $u(a) = -\alpha/(1-\alpha)$.

First, we show u is order preserving. If $a > b$, then there are several cases depending upon their location relative to a^* and b^*. We consider two typical cases. If $a > a^* \gtrsim b \gtrsim b^*$, then $u(a) > 1 \geq u(b)$. If $a^* > a > b > b^*$, then by the Archimedean condition there exist $\alpha, \beta \varepsilon (0,1)$ such that $a \backsim a^*\alpha b^*$, $b \backsim a^*\beta b^*$; and from above, $u(a) = \alpha$ and $u(b) = \beta$. If $\alpha \leq \beta$, then, since $P(a^*,b^*) \neq P(b^*,a^*)$, the consistency of P (theorem 12) implies $a \backsim a^*\alpha b^* \lesssim a^*\beta b^* \backsim b$, contrary to hypothesis. Thus, $u(a) > u(b)$.

Second, we show u is linear. Again one must consider a number of cases, of which we will consider only two. If $a^* > a, b > b^*$, then by the Archimedean condition there exist $\alpha, \beta \varepsilon (0,1)$ such that $a \backsim a^*\alpha b^*$ and $b \backsim a^*\beta b^*$. Using the axioms of a risk space it is not difficult to show $a\lambda b \backsim a^*[\alpha\lambda + (1-\lambda)\beta]b^*$, so $u(a\lambda b) = \lambda\alpha + (1-\lambda)\beta = \lambda u(a) + (1-\lambda)u(b)$. As a second example, suppose $a > a\lambda b > a^* > b > b^*$. Then there exist $\alpha, \beta, \sigma \varepsilon (0,1)$ such that $a^* \backsim a\alpha b^*$, $b \backsim a^*\beta b^*$, and $a^* \backsim (a\lambda b)\sigma b^*$. Again using the axioms of a risk space, $b \backsim a\alpha\beta b^*$, so $a^* \backsim (a\lambda b)\sigma b^* \backsim a\rho b^*$, where $\rho = (\lambda + \alpha\beta - \lambda\alpha\beta)\sigma$. Using the decomposition assumption,

$$\begin{aligned} P(a^*, a^*) &= P(a\rho b^*, a\alpha b^*) \\ &= P(a, b^*)Q(\rho-\alpha) + P(b^*, a)Q(\alpha-\rho) \\ &= P(a\alpha b^*, a\rho b^*) \\ &= P(a, b^*)Q(\alpha-\rho) + P(b^*, a)Q(\rho-\alpha) \end{aligned}$$

Collecting terms,

$$0 = [P(a, b^*) - P(b^*, a)][Q(\rho-\alpha) - Q(\alpha-\rho)].$$

Since, $u(a) = 1/\alpha > 0$, the first term is not zero, so the second one must be. Since Q is continuous, this implies $\alpha = \rho$. Rewriting,

$$\begin{aligned} u(a\lambda b) &= 1/\sigma = \lambda/\alpha + (1-\lambda)\beta \\ &= \lambda u(a) + (1-\lambda)u(b). \end{aligned}$$

Next, we establish the form of P. As above, there are a number of cases to consider and we will only examine one, namely, $a > a^* > b > b^*$. Let $\lambda = u(a) - u(b)$. By the Archimedean condition, there exist $\alpha, \beta \varepsilon (0,1)$ such that $a^* \backsim a\alpha b^*$ and $b \backsim a^*\beta b^*$, so $u(a) - u(b) = \frac{1}{\alpha} - \beta = \lambda$. Also, $b \backsim a\alpha\beta b^*$. Using these facts and the decomposition assumption, it is easy to show

$$P(a, b) - P(a^*, b^*) = P(a, b^*)[Q(\alpha\lambda) - Q(\alpha)] + P(b^*, a)[Q(-\alpha\lambda) - Q(-\alpha)].$$

If P is additive, then the additivity of Q and its functional equation imply that:

$$Q(\alpha\lambda)-Q(\alpha) = -Q(-\lambda)[Q(\alpha)-Q(-\alpha)]$$
$$= -[Q(-\alpha\lambda)-Q(-\alpha)].$$

Substituting this plus the easily verified fact that

$$P(a^*,b^*)-P(b^*,a^*) = [P(a,b^*)-P(b^*,a)][Q(\alpha)-Q(-\alpha)]$$

yields,

$$P(a,b) = P(a^*,b^*) - [P(a^*,b^*)-P(b^*,a^*)]Q(\lambda).$$

Substituting the known form for a continuous core yields the result. If P is transitive, the argument is similar and a little simpler.

The converse is immediate if one simply sets up the two sides of the decomposition equation, substitutes the given form of P and uses the linearity of u.

The final theorem is almost, but not quite, the converse of the important corollary to theorem 8, which states that any linear utility function of a decomposable linear structure is a sensation scale. To get a converse one has to suppose that P is strictly monotonic and that the core is continuous.

<u>Theorem 15</u>. <u>Let</u> P <u>be a decomposable linear structure with a continuous core</u>. <u>If</u> P <u>has a sensation scale</u> u <u>such that</u> P <u>is strictly monotonic increasing in that scale, then</u> u <u>is a linear utility function</u>.

Proof. First, we show that P is Archimedean. Suppose $a > b > c$, then by definition, $P(a,c) \geq P(b,c) \geq P(c,c)$, but by the strict monotonicity of P in u these must be strict inequalities. Since P is decomposable, it is either transitive or additive. If it is transitive, then for any $\alpha\varepsilon[0,1]$, $P(a\alpha c,c) = P(a,c)Q(\alpha)$. By assumption, Q is continuous, so by theorem 7 its range is $[0,1]$, hence $P(a\alpha c, c)$ ranges continuously from $0 = P(c,c)$ to $P(a,b)$. Therefore, there exists an $\alpha\epsilon(0,1)$ such that $P(a\alpha c, c) = P(b,c)$. A similar, but slightly more complicated argument establishes the same conclusion for the additive case. By the fact that u is a sensation scale, $P[u(a\alpha c)-u(c)] = P[u(b)-u(c)]$, so by the strict montonicity of P, $u(a\alpha c) - u(c) = u(b) - u(c)$. Thus $b \sim a\alpha c$.

Since P is Archimedean and Q is continuous, theorem 14 establishes that there is a linear utility function u'. Select some $a^* \epsilon S$ and let u and u' be those linear transformations such that $u(a^*) = u'(a^*) = 0$. Since u and u' are each utility functions, there is some strictly increasing function f such that $u = f(u')$. By an argument similar to that used above, the continuity of Q implies the continuity of f. By the corollary to theorem 8, u' as well as u is a sensation scale. Thus, if we consider $a\epsilon S$ and real δ such that $u'(a) + \delta$ is defined as the image of an element $b\epsilon S$, then

$$P([illegible]) = P[u'(a)+\delta, u'(a)] = P(\delta),$$

so by definition of f,

$$P(b,a) = P\left\{f[u'(a)+\delta], f[u'(a)]\right\} = P\left\{f[u'(a)+\delta]-f[u'(a)]\right\}.$$

By the monotonicity of P in v these equations imply there is a function g such that,

$$f[u'(a)+\delta] = f[u'(a)] + g(\delta).$$

Let $a = a^*$, then $u(a^*) = 0 = f[u'(a^*)] = f(0)$, so $f(\delta) = g(\delta)$. Thus, $f(u' + \delta) = f(u') = f(\delta)$ for all u' and δ such that they and $u' + \delta$ are images of points of S under u'. Since u' is linear, there are a continuum of such values. This, with the continuity of f, implies $f(x) = kx$. Hence, u is linear.

[illegible] applications of [illegible] utility [illegible] for [illegible]. This is known as the problem of interpersonal comparisons of utility. Since we will confine our attention to linear utility functions (which are unique up to a linear transformation), the problem amounts to selecting a general unit in terms of which to measure utility. Since utility differences, not absolute utility, are important, the zero of the scale is of no matter.

Once the utility notion is cast in a discrimination framework, and its close relation to the psychophysical concept of a just noticeable difference (jnd) is noted, one is led immediately to consider the jnd as a possible unit. For example, in a context of the social welfare function problem and under certain very special assumptions, Goodman and Markowitz [2] employed this idea. In general utility theory, the idea certainly makes no sense at all until it is known that a linear utility function is a sensation scale, so that the jnd is constant; this we have shown for decomposable linear structures in theorem 8.

There are two further aspects of this idea which must be considered before it can be considered a sensible resolution of the interpersonal comparison problem. One — a purely technical matter within the framework we have given — is the question whether the comparison effected by the jnd unit is independent of the probability cut-off (see section 3) used to define the semiorder from which the jnd function arises. This we shall examine below.

Assuming that the technical question can be resolved favorably, the second question is whether equating jnds between people does in fact solve the interpersonal comparison problem. Certainly, many people with whom I have spoken are unconvinced that it does solve it; at the very least, they feel there is not now any convincing argument. With this I agree, but intuitively I feel that equating jnds is roughly what one means by such a comparison. A person considers how sensitive he is to a change, which he measures subjectively in terms of the number of his just discernable units, and he compares this with how sensitive another person is to the change in terms of that person's jnd. It seems to me that when one makes such comparisons one is invariably invoking a comparison of sensitivity, and I would argue that "sensitivity" means that which is "just discriminated." One measure of this is the jnd. Thus, equating jnds is suggested as the solution to the interpersonal comparison problem, provided that the technical problem mentioned before can be resolved. As we shall see, this leads to an empirical question, the answer to which is not now known. Whether defining jnd units to be equal does capture the idea of interpersonal comparison seems at present to be a difficult empirical problem.

Now, to the technical question. Suppose that an arbitrary probability cutoff k, $\frac{1}{2} \leq k < 1$, is selected, and suppose that we have a decomposable linear structure P with a continuous core Q and a linear utility function u. We wish to find an expression for the jnd functions of u corresponding to the induced semiorder (definition 5). These are constant except at the bounds of the utility function.

Let $a^*, b^* \epsilon S$ be the elements described in axiom D3, and suppose $a^* > b^*$. We may take the linear transformation of the utility function so that $u(a^*) = 1$ and $u(b^*) = 0$. By the remarks following theorem 4 of [4] we know that the jnd function δ can be expressed as

$$\begin{aligned}\delta &= \overline{\alpha}(a^*, b^*) \\ &= \sup[\alpha \mid (a^* \alpha b^*) I b^*]\end{aligned}$$

By definition 5 and the fact that P is regular (theorem 6), $(a^* \alpha b^*) I b^*$ if and only if

$$\begin{aligned}k &\geq P(a^* \alpha b^*, b^*) \\ &= P(a^*, b^*) Q(\alpha) + P(b^*, a^*) Q(-\alpha).\end{aligned}$$

In the additive case, this reduces to

$$k \geq \frac{1}{2}\left\{K + [P(a^*, b^*) - P(b^*, a^*)]\alpha^{\epsilon}\right\},$$

and in the transitive case to

$$k \geq P(a^*, b^*)\alpha^{\epsilon} .$$

Solving,

$$\delta = \left[\frac{2k-K}{P(a^*, b^*) - P(b^*, a^*)}\right]^{1/\epsilon} \text{, if P is additive,}$$

and

$$\delta = \left[\frac{k}{P(a^*, b^*)}\right]^{1/\epsilon} \text{, if P is transitive}$$

Now, suppose that we have two different people, who under the same conditions express preferences between alternatives of the same risk space. Suppose that each person yields a decomposable linear structure having a linear utility function and a continuous core; call them P_1 and P_2. Since the same experimental conditions obtain, we may suppose that both structures are additive or both are transitive.

For each person, the same probability cutoff k is used, but a*, and b* may differ. While a priori, the values of K may also differ, in practice forced answers probably would be used in which case K = 1. This we will assume. Then the ratio of the jnds is given by

$$\frac{\delta_1}{\delta_2} = \frac{[P(a^*_2,b^*_2)-P(b^*_2,a^*_2)]^{1/\varepsilon_2}}{[P(a^*_1,b^*_1)-P(b^*_1,a^*_1)]^{1/\varepsilon_1}} (2k-1)^{\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2}}, \text{ if P is additive}$$

and

$$\frac{\delta_1}{\delta_2} = \frac{[P(a^*_2,b^*_2)]^{1/\varepsilon_2}}{[P(a^*_1,b^*_1)]^{1/\varepsilon_1}} (k)^{\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2}}, \text{ if P is transitive.}$$

Thus, we see that the above definition for resolving interpersonal comparisons makes no sense at all unless $\varepsilon_1 = \varepsilon_2$, for otherwise it depends upon the parameter k, which is an artifact of our calculations. Assuming the decomposition assumption is met, it is an empirical question whether the parameter ε is (approximately) constant over the population. If so, this would be an extremely interesting result regardless of the interpersonal comparison problem. Note, the constancy of ε would not imply that all people have identical discrimination functions over events in the world, but it would mean this over subjective probabilities (see section 7).

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